

Higher spin fermions in the BTZ black hole

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ABSTRACT: Recently it has been shown that the wave equations of bosonic higher spin fields in the BTZ background can be solved exactly. In this work we extend this analysis to fermionic higher spin fields. We solve the wave equations for arbitrary half-integer spin fields in the BTZ black hole background and obtain exact expressions for their quasinormal modes. These quasinormal modes are shown to agree precisely with the poles of the corresponding two point function in the dual conformal field theory as predicted by the AdS/CFT correspondence. We also obtain an expression for the 1-loop determinant in terms of the quasinormal modes and show it agrees with that obtained by integrating the heat kernel found by group theoretic methods.

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1. Introduction

Studying the behaviour of various fields in black hole backgrounds have always revealed useful information of the nature of black holes. Solving the equation of motion of various fields in the black hole background is usually the first step in understanding properties like Hawking radiation and quasinormal modes of the black hole. In most situations the equations of motion are solvable only numerically or in some limits. The BTZ black hole is a rare example of a black hole background in which the equations of motion of all integer spins are solvable in closed form [1]. In terms of hypergeometric functions. In [1] the complete spectrum of quasinormal modes of all integer spin fields in the BTZ background was obtained and they were shown to agree with the prediction from AdS_3/CFT_2 . The aim of this paper is to generalize this result to the case of arbitrary half integer spins. Thus the BTZ black hole is perhaps the only example of a black hole in which spectrum of excitations of arbitrary spin fields can be completely solved in closed form. In fact classical string propagation in the BTZ background is shown to be integrable [2]. All these observations suggest that it is possible to obtain the complete spectrum of string excitations.

Arbitrary massive half integer spins in 3 dimensional backgrounds are present in all examples of AdS_3/CFT_2 that occur in string theory. A prototype example of this is the D1-D5 system and the corresponding BTZ black hole [3]. Arbitrary massless half integer higher spins also occur in the supersymmetric versions of the AdS_3/CFT_2 duality proposed by Gaberdiel and Gopakumar [4] which has recently been studied in [5]. Let us briefly recall the AdS_3/CFT_2 dictionary regarding a field of spin s . AdS_3/CFT_2 duality relates a spin s field propagating in AdS_3 to an operator \mathcal{O} in the dual conformal field theory characterized by conformal weights (h_L, h_R) with [6]

$$h_R - h_L = \pm s. \quad (1.1)$$

The mass of the propagating field m is related to the conformal dimension of the operator \mathcal{O} which is given by

$$h_R + h_L = \hat{\Delta}. \quad (1.2)$$

When the conformal field theory is at finite temperature it is dual to the BTZ black hole. Then the poles of the retarded two point function of the operator \mathcal{O} are given by the quasinormal modes of the spin s field [7, 8, 9]. The two point function of the operator \mathcal{O} is determined by conformal invariance. The poles of the retarded Green's function in the complex frequency plane are then given by [10, 7]

$$\omega_L = k - 4\pi i T_L (\hat{n} + h_L), \quad \omega_R = -k - 4\pi i T_R (\hat{n} + h_R). \quad (1.3)$$

ω and k refer to the frequency and momentum respectively and $\hat{n} = 0, 1, 2, \dots$. The L, R subscripts denote left and right moving poles T_L, T_R are the left and right moving temperatures of the CFT which are related to the temperature and the angular potential of the BTZ black hole. In [1] we have verified this prediction for arbitrary integer spins by explicitly solving the wave equations of the higher spin field and obtaining their quasi-normal modes. In this paper we solve the wave equations of arbitrary half integer spins in the BTZ background and obtain their quasinormal modes. It will be shown that the quasinormal modes coincide with the location of the poles predicted by the AdS_3/CFT_2 correspondence given in (1.3). Thus this work completes the analysis begun in [1]. It is indeed remarkable that the wave equations of arbitrary spins can be solved in closed form for the BTZ background.

An interesting property of quasi-normal modes discovered recently, is that the 1-loop determinant of the corresponding field can be obtained by considering suitable products of the quasi-normal modes [11]. We show that by considering suitable products of the quasi-normal modes of the half integer spin fields we reproduce the one loop determinant constructed by integrating the heat kernel in thermal AdS_3 found by group theoretic methods in [12].

The organization of this paper is as follows. We will first review the description of higher spin fermions in AdS_3 . This will enable us to introduce the notations and the conventions we use in this paper. In section 3 we will show that the equations of

motion of arbitrary half integer spin fields in the BTZ background can be simplified and solutions can be found in closed form in terms of hypergeometric functions. This generalizes the work of [13] which solves the wave equations for the $s = 1/2$ case. Once the solutions have been found we extract their quasi-normal modes and show that it agrees with that given in (1.3). In section 4 we write down the one-loop determinant for the higher spin fermions in terms of products over the corresponding quasi-normal modes and show that it agrees with that evaluated by group theoretic methods. Appendix A contains some details about the geometry of the BTZ black hole and the properties of Laplacians of higher spin fermions. Appendix B contains the proof of an identity which is required in our analysis of the wave equations for the higher spin fermions.

2. Description of higher spin fermions in AdS_3

The massive higher spin fermion fields with $s = n + \frac{1}{2}$ are realized by completely symmetric tensors of rank n , $\Psi_{\mu_1\mu_2\cdots\mu_n}$. They satisfy the following equations [14, 15].

$$[\Gamma^\mu \nabla_\mu - m_n] \Psi_{\mu_1\mu_2\cdots\mu_n} = 0, \quad (2.1)$$

$$\nabla^\mu \Phi_{\mu\mu_2\cdots\mu_n} = 0, \quad (2.2)$$

$$\Gamma^\mu \Psi_{\mu\mu_2\cdots\mu_n} = 0. \quad (2.3)$$

Here $\Psi_{\mu_1,\mu_2\cdots\mu_n}$ is a two component Dirac fermion which is totally symmetric in the indices $\mu_1, \mu_2 \cdots \mu_n$. The mass of the spin $(n + \frac{1}{2})$ field in AdS_3 is denoted as m_n . It is usually written as the following sum [14]

$$m_n = \left(n - \frac{1}{2}\right) + M. \quad (2.4)$$

where the first term is due to the curvature of AdS_3 and M is the actual mass of the field. Here we have set the radius of AdS_3 to unity. The curved space Dirac matrices obey the Clifford algebra.

$$\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}. \quad (2.5)$$

Here the Γ^μ is written using the vierbiens (e_a^μ) by $\Gamma^\mu = \gamma^a e_a^\mu$. We use the mostly + convention for the metric $g_{\mu,\nu}$ and $\mu, \nu, a \in \{0, 1, 2\}$. The covariant derivative is defined as

$$\begin{aligned} \nabla_\mu \Psi_{\mu_1\mu_2\cdots\mu_n} &= \partial_\mu \Psi_{\mu_1\mu_2\cdots\mu_n} + \frac{1}{8} \omega_\mu^{ab} [\gamma_a, \gamma_b] \Psi_{\mu_1\mu_2\cdots\mu_n} \\ &\quad - \tilde{\Gamma}_{\mu\mu_1}^\rho \Psi_{\rho,\mu_2\cdots\mu_n} - \cdots - \tilde{\Gamma}_{\mu\mu_n}^\rho \Psi_{\mu_1\cdots\rho}, \end{aligned} \quad (2.6)$$

where $\tilde{\Gamma}$ are the Christoffel symbols and ω_μ^{ab} refer to the spin connection. Equation (2.3) and (2.5) also imply

$$g^{\mu\nu} \Psi_{\mu\mu_2\cdots\mu_n} = 0. \quad (2.7)$$

One important fact of higher spin fermionic fields in AdS_3 which will be important for our analysis is that the Dirac equation (2.1) and the gauge condition (2.2) is equivalent to the following first order equation [16] together with the traceless condition (2.3) for $m \neq 0$.

$$\Gamma^{\mu\nu\rho}\nabla_\nu\Psi_{\rho\mu_2\ldots\mu_n} + m\Gamma^{\mu\nu}\Psi_{\nu\mu_2\ldots\mu_n} = 0, \quad (2.8)$$

where

$$\begin{aligned} \Gamma^{\mu\nu} &= \frac{1}{2}(\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu), \\ &= \Gamma^\mu\Gamma^\nu - g^{\mu\nu}. \end{aligned} \quad (2.9)$$

The second line of the above equation is obtained by using the relation in (2.5). Similarly

$$\begin{aligned} \Gamma^{\mu\nu\rho} &= \frac{1}{3!}(\Gamma^\mu\Gamma^\nu\Gamma^\rho - \Gamma^\nu\Gamma^\mu\Gamma^\rho + \text{et. cycl.}), \\ &= \Gamma^\mu\Gamma^\nu\Gamma^\rho - g^{\mu\nu}\Gamma^\rho - g^{\nu\rho}\Gamma^\mu + g^{\mu\rho}\Gamma^\nu. \end{aligned} \quad (2.10)$$

where again we have repeatedly used (2.5) to obtain the second line. Note that for $n = 1$ the equation in (2.8) reduces to the Rarita-Schwinger equation for the gravitino. We shall now show that the covariant gauge condition (2.2) is automatically implied once we have (2.8) and (2.3). Using (2.10) in (2.8) and contracting with Γ_μ we obtain

$$\Gamma_\mu(\Gamma^\mu\Gamma^\nu\Gamma^\rho - g^{\mu\nu}\Gamma^\rho - g^{\nu\rho}\Gamma^\mu + g^{\mu\rho}\Gamma^\nu)\nabla_\nu\Psi_{\rho\mu_2\ldots\mu_n} + m\Gamma_\mu(\Gamma^\mu\Gamma^\nu - g^{\mu\nu})\Psi_{\nu\mu_2\ldots\mu_n} = 0. \quad (2.11)$$

Using the fact $\Gamma_\mu\Gamma^\mu = 3$ we get

$$[\nabla(\Gamma^\rho\Psi_{\rho\mu_2\ldots\mu_n}) + \nabla^\rho\Psi_{\rho\mu_2\ldots\mu_n}] + 2m\Gamma^\nu\Psi_{\nu\mu_2\ldots\mu_n} = 0. \quad (2.12)$$

Imposing the tracelessness condition (2.3) results in

$$\nabla^\rho\Psi_{\rho\mu_2\ldots\mu_n} = 0, \quad (2.13)$$

for $m \neq 0$. We will now show that the equation (2.8) is equivalent to (2.1) when we have tracelessness (2.3) and the gauge condition (2.13). Using (2.10) in (2.8) we obtain,

$$(\Gamma^\mu\Gamma^\nu\Gamma^\rho - g^{\mu\nu}\Gamma^\rho - g^{\nu\rho}\Gamma^\mu + g^{\mu\rho}\Gamma^\nu)\nabla_\nu\Psi_{\rho\mu_2\ldots\mu_n} + m(\Gamma^\mu\Gamma^\nu - g^{\mu\nu})\Psi_{\nu\mu_2\ldots\mu_n} = 0. \quad (2.14)$$

In the first pair of parentheses, the first three terms do not contribute because of (2.3) and (2.13) respectively. Similarly the first term in the second pair of parentheses do not contribute due to (2.3). Thus on multiplying by $g_{\mu\mu_1}$ we obtain

$$\nabla\Psi_{\mu_1\mu_2\ldots\mu_n} - m\Psi_{\mu_1\mu_2\ldots\mu_n} = 0. \quad (2.15)$$

This is clearly equivalent to (2.1) with the following relation between masses.

$$m = m_n. \quad (2.16)$$

For definiteness we will take $m > 0$ and it will be seen subsequently that this leads to the situation with $h_L - h_R = s$. The analysis can be carried out for the case with $m < 0$ and it will lead to the situation with $h_L - h_R = -s$.

Note that using (2.9) and (2.3) the Chern-Simons like equation in (2.8) can also be written as

$$g_{\mu\sigma} \Gamma^{\mu\nu\rho} \nabla_\nu \Psi_{\rho\mu_2\dots\mu_n} - m \Psi_{\sigma\mu_2\dots\mu_n} = 0. \quad (2.17)$$

As a final consistency check we show that (2.17) agrees with the symmetry of the tensor. The above equation must be symmetric under the exchange of σ and μ_2 . This equivalent to saying

$$\epsilon^{\sigma\mu_2\eta} g_{\mu\sigma} \Gamma^{\mu\nu\rho} \nabla_\nu \Psi_{\rho\mu_2\dots\mu_n} = 0, \quad (2.18)$$

should be true. Let us examine the RHS of the above equation

$$\begin{aligned} \epsilon^{\sigma\mu_2\eta} g_{\mu\sigma} \Gamma^{\mu\nu\rho} \nabla_\nu \Psi_{\rho\mu_2\dots\mu_n} &= \epsilon_\mu^{\mu_2\eta} \Gamma^{\mu\nu\rho} \nabla_\nu \Psi_{\rho\mu_2\dots\mu_n}, \\ &= \epsilon_\mu^{\mu_2\eta} g^{\mu\rho} \nabla_\nu \Psi_{\rho\mu_2\dots\mu_n}, \\ &= \epsilon^{\rho\mu_2\eta} \nabla_\nu \Psi_{\rho\mu_2\dots\mu_n}. \\ &= 0 \end{aligned}$$

In the second step we have used similar manipulations as those used in simplifying (2.14).

The BTZ black hole is obtained by identifications of AdS_3 [17]. Thus it is locally AdS_3 and the above analysis for the equations of motion for the fermionic higher spin fields in AdS_3 can be carried over to the BTZ background. We will use the following metric for the BTZ black hole

$$ds^2 = d\xi^2 - \sinh^2 \xi dx_+^2 + \cosh^2 \xi dx_-^2. \quad (2.19)$$

The horizon is at $\xi = 0$ and the boundary is at $\xi = \infty$. The relation between these co-ordinates and the conventional co-ordinates is reviewed in Appendix A. The appendix also lists other properties of this background which will be used in the paper. We will use the following convention for the flat space Dirac matrices [13]

$$\gamma^0 = i\sigma^2, \quad \gamma^1 = \sigma^1, \quad \gamma^2 = \sigma^3. \quad (2.20)$$

where σ 's refer to the Pauli matrices. The vierbein for the BTZ metric in (2.19) is given by

$$e_0 = -\sinh \xi dx_+, \quad e_1 = d\xi, \quad e_2 = \cosh \xi dx_-. \quad (2.21)$$

The spin connection is given by

$$\begin{aligned}\hat{\omega}_+ &= \frac{1}{8}\omega_+^{ab}[\gamma_a, \gamma_b] = -\frac{1}{2}\cosh\xi\sigma^3, \\ \hat{\omega}_- &= \frac{1}{8}\omega_-^{ab}[\gamma_a, \gamma_b] = \frac{i}{2}\sinh\xi\sigma^2, \\ \omega_\xi^{ab} &= 0.\end{aligned}\tag{2.22}$$

3. Solving higher spin fermion equations

In this section we solve the fermionic higher spin equations (2.1) in the background of the BTZ black hole and obtain their quasi-normal modes. Our strategy will be the following: Note that due to the traceless condition (2.3) we can restrict our attention to components of the higher spin fermion whose indices lie along the $+$ and $-$ directions. The tracelessness condition (2.3) for the BTZ metric in (2.19) is given by

$$\gamma^0 \frac{1}{\sinh\xi} \Psi_{+\mu_2\dots\mu_n} + \gamma^1 \Psi_{\xi\mu_2\dots\mu_n} + \gamma^2 \frac{1}{\cosh\xi} \Psi_{-\mu_2\dots\mu_n} = 0,\tag{3.1}$$

from which we obtain

$$\Psi_{\xi\mu_2\dots\mu_n} = -\gamma^1\gamma^0 \frac{1}{\sinh\xi} \Psi_{+\mu_2\dots\mu_n} - \gamma^1\gamma^2 \frac{1}{\cosh\xi} \Psi_{-\mu_2\dots\mu_n}.\tag{3.2}$$

From the above equation it is clear any component with whose indices lie along the radial coordinate ξ can be expressed in terms of components along $+$ and $-$ directions by the repeated use of (3.2). The next step in our analysis which is carried out in section 3.1 consists of showing that the spin- $(n + \frac{1}{2})$ Dirac operator on components with indices only along $+$ and $-$ directions reduces to that of the spin $\frac{1}{2}$ Dirac operator but with a mixing ‘mass matrix’. This property is similar to the observation seen for the bosonic higher spin fields in [1] where the spin s Laplacian reduced to the scalar Laplacian with a mixing mass matrix. In section 3.2 the equations are diagonalized and solved in terms of hypergeometric functions up to a constant which depends on the polarization of the higher spin component. We then show that all the polarizations can be related to a single constant using (2.17). In section 3.3 we determine the quasi-normal modes and show that they coincide with the poles of the corresponding two point function as expected from the AdS_3/CFT_2 correspondence.

3.1 Reducing the higher spin Dirac operator to the spin $\frac{1}{2}$ Dirac operator

The action of the Dirac operator ∇ on $\Psi_{\mu_1\mu_2\ldots\mu_p\nu_1\nu_2\ldots\nu_q}$ is given by

$$\begin{aligned} \nabla \Psi_{\mu_1\mu_2\ldots\mu_p\nu_1\nu_2\ldots\nu_q} &= \Gamma^\mu (\partial_\mu + \omega_\mu) \Psi_{\mu_1\mu_2\ldots\mu_p\nu_1\nu_2\ldots\nu_q} \\ &\quad - \Gamma^\mu (\tilde{\Gamma}_{\mu\mu_1}^\eta \Psi_{\eta\mu_2\ldots\mu_p\nu_1\nu_2\ldots\nu_q} + \tilde{\Gamma}_{\mu\mu_2}^\eta \Psi_{\mu_1\eta\mu_3\ldots\mu_p\nu_1\nu_2\ldots\nu_q} + \cdots + \tilde{\Gamma}_{\mu\mu_p}^\eta \Psi_{\mu_1\ldots\mu_{p-1}\eta\nu_1\nu_2\ldots\nu_q} \\ &\quad + \tilde{\Gamma}_{\mu\nu_1}^\eta \Psi_{\mu_1\mu_2\ldots\mu_p\eta\nu_2\ldots\nu_q} + \tilde{\Gamma}_{\mu\nu_2}^\eta \Psi_{\mu_1\mu_2\ldots\mu_p\nu_1\eta\nu_3\ldots\nu_q} + \cdots + \tilde{\Gamma}_{\mu\nu_q}^\eta \Psi_{\mu_1\ldots\mu_p\nu_1\nu_2\ldots\nu_{q-1}\eta}). \end{aligned} \quad (3.3)$$

As we have argued earlier it is sufficient to examine the components with indices along the $+$ and $-$ coordinates. Therefore we choose $\mu_1 \cdots \mu_p = + \cdots +$ and $\nu_1 \cdots \nu_q = - \cdots -$. We also introduce the notation ' (p) ' to mean p number of $+$ indices and by we mean ' (q) ' number of $-$ indices with $p + q = n$. For example consider a spin- $\frac{11}{2}$ field with $n = 5$ then

$$\Psi_{(2)(3)} = \Psi_{++----}. \quad (3.4)$$

Note that the operator occurring in the first term of (3.3) is same as the Dirac operator ∇ acting on the spin- $\frac{1}{2}$ object Ψ . We denote this as $\Delta \Psi_{(p)(q)}$. Let us define

$$\begin{aligned} \mathcal{A}_{\mu(p)(q)} &= \tilde{\Gamma}_{\mu\mu_1}^\eta \Psi_{\eta\mu_2\ldots\mu_p\nu_1\nu_2\ldots\nu_q} + \tilde{\Gamma}_{\mu\mu_2}^\eta \Psi_{\mu_1\eta\mu_3\ldots\mu_p\nu_1\nu_2\ldots\nu_q} + \cdots + \tilde{\Gamma}_{\mu\mu_p}^\eta \Psi_{\mu_1\ldots\mu_{p-1}\eta\nu_1\nu_2\ldots\nu_q} \\ &\quad + \tilde{\Gamma}_{\mu\nu_1}^\eta \Psi_{\mu_1\mu_2\ldots\mu_p\eta\nu_2\ldots\nu_q} + \tilde{\Gamma}_{\mu\nu_2}^\eta \Psi_{\mu_1\mu_2\ldots\mu_p\nu_1\eta\nu_3\ldots\nu_q} + \cdots + \tilde{\Gamma}_{\mu\nu_q}^\eta \Psi_{\mu_1\ldots\mu_p\nu_1\nu_2\ldots\nu_{q-1}\eta} \\ &= p \tilde{\Gamma}_{\mu+}^\eta \Psi_{\eta(p-1)(q)} + q \tilde{\Gamma}_{\mu-}^\eta \Psi_{\eta(p)(q-1)}. \end{aligned} \quad (3.5)$$

Thus with these notations and definitions we have

$$\nabla \Psi_{(p)(q)} = \Delta \Psi_{(p)(q)} - \Gamma^\mu \mathcal{A}_{\mu(p)(q)}. \quad (3.6)$$

where the first term is the ordinary Dirac operator and the second term of (3.6) is given by

$$\begin{aligned} \Gamma^\mu \mathcal{A}_{\mu(p)(q)} &= \gamma^a e_a^\mu \mathcal{A}_{\mu(p)(q)}, \\ &= \gamma^0 e_0^+ \mathcal{A}_{+(p)(q)} + \gamma^1 e_1^\xi \mathcal{A}_{\xi(p)(q)} + \gamma^2 e_2^- \mathcal{A}_{-(p)(q)}, \\ &= \gamma^1 \mathcal{A}_{\xi(p)(q)} + \gamma^0 \frac{1}{\sinh \xi} \mathcal{A}_{+(p)(q)} + \gamma^2 \frac{1}{\cosh \xi} \mathcal{A}_{-(p)(q)}. \end{aligned} \quad (3.7)$$

We will now evaluate each of the components of \mathcal{A} 's explicitly: We begin with the component $\mathcal{A}_{\xi(p)(q)}$ which is given by

$$\begin{aligned} \mathcal{A}_{\xi(p)(q)} &= p \tilde{\Gamma}_{\xi+}^+ \Psi_{+(p-1)(q)} + q \tilde{\Gamma}_{\xi-}^- \Psi_{-(p)(q-1)}, \\ &= (p \coth \xi + q \tanh \xi) \Psi_{(p)(q)}. \end{aligned} \quad (3.8)$$

Evaluating $\mathcal{A}_{+(p)(q)}$ leads to

$$\begin{aligned}
\mathcal{A}_{+(p)(q)} &= p\tilde{\Gamma}_{++}^{\eta}\Psi_{\eta(p-1)(q)} + q\tilde{\Gamma}_{+-}^{\eta}\Psi_{\eta(p)(q-1)}, \\
&= p\tilde{\Gamma}_{++}^{\xi}\Psi_{\xi(p-1)(q)} = p \cosh \xi \sinh \xi \Psi_{\xi(p-1)(q)}, \\
&= p \cosh \xi \sinh \xi (-\gamma^1\gamma^0 \frac{1}{\sinh \xi} \Psi_{+(p-1)(q)} - \gamma^1\gamma^2 \frac{1}{\cosh \xi} \Psi_{-(p-1)(q)}), \\
&= -p\gamma^1\gamma^0 \cosh \xi \Psi_{(p)(q)} - p\gamma^1\gamma^2 \sinh \xi \Psi_{(p-1)(q+1)}. \tag{3.9}
\end{aligned}$$

To obtain the last line in the above equation we have used the condition (3.2). Finally $\mathcal{A}_{-(p)(q)}$ is given by

$$\begin{aligned}
\mathcal{A}_{-(p)(q)} &= p\hat{\Gamma}_{-+}^{\eta}\Psi_{\eta(p-1)(q)} + q\hat{\Gamma}_{--}^{\eta}\Psi_{\eta(p)(q-1)}, \\
&= q\hat{\Gamma}_{--}^{\xi}\Psi_{\xi(p)(q-1)} = -q \cosh \xi \sinh \xi \Psi_{\xi(p)(q-1)}, \\
&= -q \cosh \xi \sinh \xi (-\gamma^1\gamma^0 \frac{1}{\sinh \xi} \Psi_{+(p)(q-1)} - \gamma^1\gamma^2 \frac{1}{\cosh \xi} \Psi_{-(p)(q-1)}), \\
&= q\gamma^1\gamma^0 \cosh \xi \Psi_{(p+1)(q-1)} + q\gamma^1\gamma^2 \sinh \xi \Psi_{(p)(q)}. \tag{3.10}
\end{aligned}$$

Again we have used the condition (3.2) to obtain the last line. Substituting (3.8), (3.9) and (3.10) in (3.7) and using $\gamma^0\gamma^1\gamma^2 = 1 = -\gamma^2\gamma^1\gamma^0$ we obtain.

$$\Gamma^{\mu}\mathcal{A}_{\mu(p)(q)} = -p\Psi_{(p-1)(q+1)} - q\Psi_{(p+1)(q-1)}. \tag{3.11}$$

Thus equation (3.6) reduces to

$$\begin{aligned}
\nabla\Psi_{(p)(q)} &= \Delta\Psi_{(p)(q)} + p\Psi_{(p-1)(q+1)} + q\Psi_{(p+1)(q-1)}, \\
\text{or } \nabla\Psi_{(p)} &= \Delta\Psi_{(p)} + p\Psi_{(p-1)} + (s-p)\Psi_{(p+1)}. \tag{3.12}
\end{aligned}$$

In the second line we have suppressed the label (q) with the understanding that we will always have $p+q=n$. Thus we have reduced the action of the higher spin Dirac operator to that of the ordinary Dirac operator together with a mixing ‘mass matrix’.

3.2 Solutions of the spin- $(n+\frac{1}{2})$ components

Substituting (3.12) into the Dirac equation equation (2.15) we obtain

$$\Delta\Psi_{(p)} + p\Psi_{(p-1)} + (s-p)\Psi_{(p+1)} - m\Psi_{(p)} = 0. \tag{3.13}$$

This can be written as

$$\Delta\Psi_{(p)} - M_{pr}^{(n)}\Psi_{(r)} = 0, \tag{3.14}$$

with the $(n+1) \times (n+1)$ matrix, M_{pr} defined as

$$M_{pr}^{(n)} = -p\delta_{p-1,r} - (s-p)\delta_{p+1,r} + m\delta_{p,r}. \tag{3.15}$$

Note that this is a closed equation for the components of the tensor with boundary indices as $+$ or $-$. It is basically $n+1$ coupled Dirac equations. In this section we will obtain the solutions for these components explicitly.

Diagonalization of the mass matrix

The first task is decouple the equations in (3.14) or in other words diagonalize the matrix $M_{pr}^{(n)}$. Following the method developed in [1] we consider the linear combination

$$\Psi_{[p]} = \sum_{a=0}^p \sum_{b=0}^{n-p} (-1)^b \binom{p}{a} \binom{n-p}{b} \Psi_{(n-a-b)}. \quad (3.16)$$

We can also express $\Psi_{[p]}$ as,

$$\Psi_{[p]} = \sum_{q=0}^s T_{pq}^{(n)} \Psi_{(q)}. \quad (3.17)$$

The transformation matrix $T_{pq}^{(n)}$ is defined as follows. Consider the polynomial,

$$\sum_{q=0}^n T_{pq}^{(n)} x^q = \sum_{a=0}^p \sum_{b=0}^{n-p} (-1)^b \binom{n}{a} \binom{n-p}{b} x^{(n-a-b)}, \quad (3.18)$$

this can be rewritten as

$$\sum_{q=0}^s T_{pq}^{(n)} x^q = (x+1)^p (x-1)^{s-p}. \quad (3.19)$$

Thus $T_{pq}^{(s)}$ is the coefficient of x^q of the function above. A formal expression for $T_{pq}^{(n)}$ can then be obtained by a Taylor series expansion and be expressed as a contour integral.

$$T_{pq}^{(n)} = \frac{1}{2\pi i} \oint \frac{dx}{x^{q+1}} (x+1)^p (x-1)^{s-p}. \quad (3.20)$$

The transformation matrix $T^{(n)}$ obeys the following identities

Identity 1 :

$$\sum_{q=0}^s T_{pq}^{(n)} T_{qr}^{(n)} = 2^n \delta_{pr}. \quad (3.21)$$

This identity has been derived in Appendix C of [1]. We also have

Identity 2 :

$$(T^{(n)} M^{(n)} [T^{(n)}]^{-1})_{pq} = (n - 2p + m) \delta_{pq}. \quad (3.22)$$

i.e., $T^{(n)} M^{(n)} [T^{(n)}]^{-1}$ is diagonal. The proof of this identity is provided in Appendix B. Substituting for $\Psi_{(p)}$ in terms of $\Psi_{[p]}$ and using (3.22) we obtain the following set of decoupled Dirac equations

$$\Delta \Psi_{[p]} - (m + n - 2p) \Psi_{[p]} = 0. \quad (3.23)$$

Since this is just a set of Dirac equations, they can be easily solved using the solutions of [13, 7]. We first substitute the following ansatz for the two components of the spinor

$$\Psi_{[p]}^{(1,2)} = \frac{e^{-i(k_+x^+ + k_-x^-)}}{\sqrt{\cosh \xi \sinh \xi}} \psi_{[p]}^{(1,2)}(\xi). \quad (3.24)$$

in the equation (3.23). Note that with the definition of x^+ and x^- given in (A.3) we see that the frequency and momenta of these solutions are related to k_+ and k_- by

$$(k_+ + k_-)(r_+ - r_-) = \omega - k, \quad (k_+ - k_-)(r_+ + r_-) = \omega + k. \quad (3.25)$$

Substituting (3.24) into the decoupled Dirac equation (3.23) reduces the equation to

$$\gamma^1 \partial_\xi \psi_{[p]} - \gamma^0 \frac{ik_+}{\sinh \xi} \psi_{[p]} - \gamma^2 \frac{ik_-}{\cosh \xi} \psi_{[p]} - (m + n - 2p) \psi_{[p]} = 0. \quad (3.26)$$

We then substitute

$$\psi_{[p]}^\pm = \psi_{[p]}^{(1)} \pm \psi_{[p]}^{(2)} = (1 - \tanh^2 \xi)^{-1/4} \sqrt{1 \pm \tanh \xi} (\psi_{[p]}^{\prime(1)} \pm \psi_{[p]}^{\prime(2)}). \quad (3.27)$$

in (3.26) which leads to the following equations

$$\begin{aligned} 2\sqrt{z}(1-z)\partial_z \psi_{[p]}^{\prime(1)} + i \left(\frac{k_+}{\sqrt{z}} + k_- \sqrt{z} \right) \psi_{[p]}^{\prime(1)} &= - \left[i(k_+ + k_-) - m - n + 2p + \frac{1}{2} \right] \psi_{[p]}^{\prime(2)}, \\ 2\sqrt{z}(1-z)\partial_z \psi_{[p]}^{\prime(2)} - i \left(\frac{k_+}{\sqrt{z}} + k_- \sqrt{z} \right) \psi_{[p]}^{\prime(2)} &= - \left[-i(k_+ + k_-) - m - n + 2p + \frac{1}{2} \right] \psi_{[p]}^{\prime(1)}. \end{aligned} \quad (3.28)$$

The solutions of these equations which obey the ingoing boundary conditions at the horizon are [7]

$$\begin{aligned} \psi_{[p]}^{\prime(1)} &= d_{[p]} z^\alpha (1-z)^{\beta_p} F(a_{[p]}, b_{[p]}, c_{[p]}; z), \\ \psi_{[p]}^{\prime(2)} &= d_{[p]} \frac{a_{[p]} - c_{[p]}}{c_{[p]}} z^{\alpha+1/2} (1-z)^{\beta_{[p]}} F(a_{[p]}, b_{[p]} + 1, c_{[p]} + 1; z). \end{aligned} \quad (3.29)$$

where $\alpha = -\frac{ik_+}{2}$, $\beta_{[p]} = \frac{1}{2}(m + n - 2p - \frac{1}{2})$, $c_{[p]} = \frac{1}{2} + 2\alpha$, and

$$a_{[p]} = \frac{k_+ - k_-}{2i} + \beta_{[p]} + \frac{1}{2}, \quad b_{[p]} = \frac{k_+ + k_-}{2i} + \beta_{[p]}. \quad (3.30)$$

Defining $e_{[p]}^{(1,2)}$ as,

$$\begin{aligned} e_{[p]}^{(1)} &= d_{[p]}, \\ e_{[p]}^{(2)} &= d_{[p]} \frac{a_{[p]} - c_{[p]}}{c_{[p]}}. \end{aligned} \quad (3.31)$$

the solutions (3.29) become

$$\begin{aligned} \psi_{[p]}^{\prime(1)} &= e_{[p]}^{(1)} z^\alpha (1-z)^{\beta_p} F(a_{[p]}, b_{[p]}, c_{[p]}; z), \\ \psi_{[p]}^{\prime(2)} &= e_{[p]}^{(2)} z^{\alpha+1/2} (1-z)^{\beta_{[p]}} F(a_{[p]}, b_{[p]} + 1, c_{[p]} + 1; z). \end{aligned} \quad (3.32)$$

Note that the two components of $\psi_{[p]}'$ are determined completely up to the constants $e_{[p]}^{(1,2)}$ which we call the polarization constants.

Behaviour of the solutions near the boundary

We shall take a look at the behaviour of the solutions near the boundary, $z \rightarrow 1$. This will enable us to fix the conformal dimension $\hat{\Delta}$ of the field.

Expanding the solutions (3.32) near $z \rightarrow 1$ and using (3.24, 3.27) we have the following

$$\Psi_{[p]}^+ \sim \mathcal{C}_1(1-z)^{\frac{1}{2}-\frac{m}{2}-\frac{n}{2}+p}, \quad (3.33)$$

$$\Psi_{[p]}^- \sim \mathcal{D}_1(1-z)^{1-\frac{m}{2}-\frac{n}{2}+p} + \mathcal{D}_2(1-z)^{\frac{1}{2}+\frac{m}{2}+\frac{n}{2}+p}. \quad (3.34)$$

Here \mathcal{C}_1 , \mathcal{D}_1 and \mathcal{D}_2 are constants. For definiteness we will take $m > 0$. Thus the most singular behaviour near the boundary is given by

$$\begin{aligned} \Psi_{i_1 \dots i_n} &\sim (1-z)^{\frac{1}{2}+\frac{m}{2}-\frac{n}{2}}, \\ &\sim \hat{z}^\delta, \quad r \rightarrow \infty, \end{aligned} \quad (3.35)$$

where

$$\delta = (1 - m - n). \quad (3.36)$$

Here we have re-written z in terms of the co-ordinate \hat{z} which is related to the radial co-ordinate r by

$$\hat{z} = \frac{1}{r}. \quad (3.37)$$

The reason for this is that asymptotically near the boundary the BTZ metric (2.19) reduces to the following AdS_3 metric

$$ds^2 = \frac{1}{\hat{z}^2}(-dt^2 + d\phi^2 + d\hat{z}^2). \quad (3.38)$$

i_1, \dots, i_n denote the boundary co-ordinates. Note that using the condition in (3.2) it is easy to see that any component involving the radial co-ordinate is suppressed compared to the boundary components as r or $\xi \rightarrow \infty$. The conformal dimension $\hat{\Delta}$ of the dual operator can then be obtained from the coupling of the boundary value of the spin s field to the corresponding operator which is given by

$$\int d^2x O^{i_1, \dots, i_n} \Psi_{i_1, \dots, i_n}. \quad (3.39)$$

From conformal invariance we obtain the following expression for the conformal dimension of the dual operator

$$\hat{\Delta} = 2 - \delta - n = 1 + m. \quad (3.40)$$

Finding the polarization constants

Our next task is to find the coefficients $e_{[p]}^{(1,2)}$. Note that from the definition of these constants in (3.31) we see that their ratio is given by

$$\frac{e_{[p]}^{(2)}}{e_{[p]}^{(1)}} = \frac{a_{[p]} - c_{[p]}}{c_{[p]}} \quad (3.41)$$

Therefore it is sufficient to determine $e_{[p]}^{(1)}$. The Chern-Simons like equations in (2.17) relate the various polarization constants. We will show that using the Chern-Simons equations and the condition (3.2) it is possible to determine all the polarization constants in terms of a single constant. Since these are constants, it is sufficient to examine the equations and the functions near the horizon. To begin let us recall the following relation

$$\Psi_{[p]}^{(1)} \pm \Psi_{[p]}^{(2)} = \sqrt{\frac{\cosh \xi \pm \sinh \xi}{\cosh \xi \sinh \xi}} (\psi_{[p]}'^{(1)} \pm \psi_{[p]}'^{(2)}) e^{-i(k_+ x^+ k_- x^-)}. \quad (3.42)$$

Near the horizon $z \rightarrow 0$ we have

$$\sqrt{\frac{\cosh \xi \pm \sinh \xi}{\cosh \xi \sinh \xi}} \simeq \frac{1}{z^{1/4}} \left(1 \pm \frac{\sqrt{z}}{2} + O(z) \right). \quad (3.43)$$

Thus the near horizon behaviours of the solutions are

$$\Psi_{[p]}^{(1)} \simeq e_{[p]}^{(1)} z^{\alpha'} e^{-i(k_+ x^+ k_- x^-)}, \quad (3.44)$$

$$\Psi_{[p]}^{(2)} \simeq \left(\frac{e_{[p]}^{(1)}}{2} + e_{[p]}^{(2)} \right) z^{\alpha'+1/2} e^{-i(k_+ x^+ k_- x^-)}. \quad (3.45)$$

here $[p] = 0, 1, 2, \dots, n$. Note that $\Psi_{(p)}^{(1,2)}$ is a linear combination of $\Psi_{[p]}^{(1,2)}$ which is given in (3.17). This implies that the behaviour of $\Psi_{(p)}^{(1,2)}$ near the horizon $z \rightarrow 0$ are given by

$$\Psi_{(p)}^{(1)} \simeq e_{(p)}^{(1)} z^{\alpha'} e^{-i(k_+ x^+ k_- x^-)}, \quad \Psi_{(p)}^{(2)} \simeq e_{(p)}^{(2)} z^{\alpha'+1/2} e^{-i(k_+ x^+ k_- x^-)}. \quad (3.46)$$

with

$$e_{[p]}^{(1)} = \sum_{q=0}^s T_{pq}^{(n)} e_{(q)}^{(1)}, \quad (3.47)$$

and a similar relation for the second component of the polarization. We will see subsequently that we will obtain closed equations for the polarization $e_{(p)}^{(1)}$ which will be sufficient to obtain all the polarization constants in terms of a single one. In order to obtain the behaviour of $\Psi_{\xi(p)(q)}^{(1,2)}$ we consider the tracelessness condition

$$\Gamma^\mu \Psi_{\mu(p)(q)} = 0. \quad (3.48)$$

This results in

$$\Psi_{\xi(p)(q)} = -\frac{1}{\sinh \xi} \gamma^1 \gamma^0 \Psi_{+(p)(q)} - \frac{1}{\cosh \xi} \gamma^1 \gamma^2 \Psi_{-(p)(q)}, \quad (3.49)$$

From examining the leading terms in the above equation near the horizon, $z \rightarrow 0$ we obtain

$$\Psi_{\xi(p)(q)} \simeq \begin{pmatrix} e_{+(p)(q)}^{(1)} z^{\alpha'-1/2} e^{-i(k_+ x^+ k_- x^-)} \\ -(e_{+(p)(q)}^{(2)} + e_{-(p)(q)}^{(1)}) z^{\alpha'} e^{-i(k_+ x^+ k_- x^-)} \end{pmatrix}. \quad (3.50)$$

Thus the behaviour of $\Psi_{\xi(p)(q)}^{(1,2)}$ near the horizon is

$$\Psi_{\xi(p)(q)}^{(1)} \simeq e_{\xi(p)(q)}^{(1)} z^{\alpha'-1/2} e^{-i(k_+ x^+ k_- x^-)}, \quad \Psi_{\xi(p)(q)}^{(2)} \simeq e_{\xi(p)(q)}^{(2)} z^{\alpha'} e^{-i(k_+ x^+ k_- x^-)}. \quad (3.51)$$

where

$$e_{\xi(p)(q)}^{(1)} = e_{+(p)(q)}^{(1)}, \quad e_{\xi(p)(q)}^{(2)} = -(e_{+(p)(q)}^{(2)} + e_{-(p)(q)}^{(1)}). \quad (3.52)$$

To obtain a closed set of equations for $e_{\pm(p)(q)}^{(1)}$ we will need to find the relation between $e_{\xi(p)(q)}^{(1)}$ and $e_{\pm(p)(q)}^{(1)}$. For this we consider the ‘ $-(p)(q)$ ’ component of the Chern-Simons equation

$$\Gamma_-^{\nu\rho} \nabla_\nu \Psi_{\rho(p)(q)} = m \Psi_{-(p)(q)}. \quad (3.53)$$

Expanding this equation and rearranging terms we obtain

$$\begin{aligned} \partial_+ \Psi_{\xi(p)(q)} - 2\sqrt{z}(1-z)\partial_z \Psi_{(p+1)(q)} + p\sqrt{z}\Psi_{(p-1)(q+2)} + q\sqrt{z}\Psi_{(p+1)(q)} \\ - \frac{1}{2\sqrt{1-z}} \sigma^{01} \Psi_{\xi(p)(q)} = m\sqrt{z}\Psi_{(p)(q+1)}. \end{aligned} \quad (3.54)$$

The tracelessness condition (2.7) was used to simplify the above equation. Near the horizon the leading terms in the above equation reduces to

$$\begin{aligned} \partial_+ \Psi_{\xi(p)(q)} - 2\sqrt{z}(1-z)\partial_z \Psi_{(p+1)(q)} + p\sqrt{z}\Psi_{(p-1)(q+2)} + q\sqrt{z}\Psi_{(p+1)(q)} \\ - \frac{1}{2} \sigma^{01} \Psi_{\xi(p)(q)} = m\sqrt{z}\Psi_{(p)(q+1)}. \end{aligned} \quad (3.55)$$

We now examine the ‘1’ component of the above spinor equation. Using (3.46), (3.51) we see that near the horizon $z \rightarrow 0$, the leading terms in the above equation is of the order $z^{\alpha-\frac{1}{2}}$. Since the equation must hold to the leading order we have the equality

$$-ik_+ e_{\xi(p)(q)}^{(1)} z^{\alpha'-1/2} - 2\sqrt{z}\partial_z (e_{(p+1)(q)}^{(1)} z^{\alpha'}) = 0. \quad (3.56)$$

This results in the following relation

$$e_{\xi(p)(q)}^{(1)} = e_{+(p)(q)}^{(1)}. \quad (3.57)$$

We now have sufficient information to find the closed set of equations for the $e_{(p)}^{(1)}$. We consider the ‘ $\xi(p)(q)$ ’ component of the Chern-Simons equation

$$g_{\xi\xi} \Gamma^{\xi+-} (\nabla_+ \Psi_{-(p)(q)} - \nabla_- \Psi_{+(p)(q)}) = m \Psi_{\xi(p)(q)}. \quad (3.58)$$

Writing this explicitly by expanding each term we have

$$\begin{aligned} \partial_+ \Psi_{-(p)(q)} - \frac{1}{2} \cosh \xi \sigma^{01} \Psi_{-(p)(q)} - \frac{p\sqrt{z}}{1-z} \Psi_{\xi-(p-1)(q)} \\ - \partial_- \Psi_{+(p)(q)} - \frac{1}{2} \sinh \xi \sigma^{21} \Psi_{+(p)(q)} - \frac{q\sqrt{z}}{1-z} \Psi_{\xi+(p)(q-1)} = -\frac{m\sqrt{z}}{1-z} \Psi_{\xi(p)(q)}. \end{aligned} \quad (3.59)$$

Examining the ‘1’ component of the equation in (3.59) near horizon $z \rightarrow 0$, we see that the leading terms go as $\sim z^\alpha$. Requiring the leading terms to satisfy the equation in (3.59) we obtain

$$\begin{aligned} -ik_+ e_{-(p)(q)}^{(1)} z^{\alpha'} - \frac{1}{2} e_{-(p)(q)}^{(1)} z^{\alpha'} - p e_{\xi(p-1)(q+1)}^{(1)} z^{\alpha'} \\ + ik_- e_{+(p)(q)}^{(1)} z^{\alpha'} - q e_{\xi(p+1)(q-1)}^{(1)} z^{\alpha'} = -m e_{\xi(p)(q)}^{(1)} z^{\alpha'}. \end{aligned} \quad (3.60)$$

Now using the (3.57) and $q = n - p - 1$ we obtain the recursion relations

$$(n - p - 1) e_{(p+2)}^{(1)} + (-m - ik_-) e_{(p+1)}^{(1)} + (ik_+ + \frac{1}{2} + p) e_{(p)}^{(1)} = 0. \quad (3.61)$$

These are a set of n equations for the $n + 1$ variables. Let us define the ‘recursion matrix’ $\tilde{C}_{ij}^{(n)}$ as

$$\tilde{C}_{jl}^{(n)} = (n - j - 1) \delta_{j+2,l} + (-m - ik_-) \delta_{j+1,l} + (j + ik_+ + \frac{1}{2}) \delta_{j,l}. \quad (3.62)$$

Note that j runs from $n - 1$ to 0 and l runs from n to 0. Thus we can write the recursion relations in (3.61) as

$$\sum_{l=0}^{n-1} \tilde{C}_{jl}^{(s)} e_{(l)}^{(1)} = 0, \quad \text{for } j = 0, 1, 2, \dots, n - 1. \quad (3.63)$$

This is same recursion relation for the polarization coefficients obtained in [1] for the higher spin bosonic case with following replacements

$$s \rightarrow n, \quad m \rightarrow -m, \quad ik_+ \rightarrow ik_+ + \frac{1}{2}. \quad (3.64)$$

From [1] we see that the recursion relation is easily solved by the change of basis to the polarization coefficients $e_{[p]}^{(1)}$. That is we have the identity

Identity 3:

$$\begin{aligned} \frac{1}{2^{n-1}} (T^{(n-1)} \tilde{C}^{(n)} T^{(n)})_{jl} \\ = (2j - n + \frac{3}{2} - m + i(k_+ - k_-)) \delta_{j+1,l} + (2j - n + \frac{1}{2} - m - i(k_+ + k_-)) \delta_{j,l}. \end{aligned} \quad (3.65)$$

The proof of this identity can be obtained using the same method as that of Identity 3 in [1] but with the replacements given in (3.64). Now performing the change of basis in the recursion relations given in (3.63) using the transformation in (3.47) we obtain

$$(2j - n + \frac{3}{2} - m + i(k_+ - k_-))e_{[j+1]}^{(1)} + (2j - n + \frac{1}{2} - m - i(k_+ + k_-))e_{[j]}^{(1)} = 0. \quad (3.66)$$

From this we get

$$e_{[n-j]}^{(1)} = -\frac{2j - n - \frac{3}{2} + m - i(k_+ - k_-)}{2j - n - \frac{1}{2} + m + i(k_+ + k_-)}e_{[n-j+1]}^{(1)}. \quad (3.67)$$

One can then write all the coefficients $e_{[n-j]}^{(1)}$ in terms of $e_{[n]}^{(1)}$.

$$e_{[n-j]}^{(1)} = (-1)^j \prod_{u=0}^{j-1} \frac{2j - n + \frac{1}{2} + m - i(k_+ - k_-)}{2j - n + \frac{3}{2} + m + i(k_+ + k_-)}e_{[n]}^{(1)}. \quad (3.68)$$

We can now solve for the polarization components $e_{[n-j]}^{(2)}$. From (3.31) which determines the ratio between the $e_{[n-j]}^{(2)}$ and $e_{[n-j]}^{(1)}$ we obtain

$$\frac{e_{[n-j]}^{(2)}}{e_{[n-j]}^{(1)}} = \frac{a_{[n-j]} - c_{[n-j]}}{c_{[n-j]}} = \frac{2j - n + m - \frac{1}{2} + i(k_+ + k_-)}{\frac{1}{2} - ik_+}. \quad (3.69)$$

Substituting (3.68) in the above equation we obtain

$$e_{[n-j]}^{(2)} = (-1)^j \prod_{u=0}^{j-1} \frac{2u - n + \frac{1}{2} + m - i(k_+ - k_-)}{2u - n - \frac{1}{2} + m + i(k_+ + k_-)}e_{[n]}^{(2)}. \quad (3.70)$$

Note that $e_{[n]}^{(2)}$ is also determined in terms of $e_{[n]}^{(1)}$ by (3.69). Thus all polarization constants are determined in terms of a single constant.

3.3 Quasinormal modes

We can now substitute the values of the polarization constants and obtain the final form of the the solutions $\psi_{[u-j]}^{(1)}$ and $\psi_{[u-j]}^{(2)}$. These are given by

$$\begin{aligned} \psi_{[n-j]}^{(1)} &= (-1)^j \prod_{u=0}^{j-1} \frac{2u - n + \frac{1}{2} + m - i(k_+ - k_-)}{2u - n + \frac{3}{2} + m + i(k_+ + k_-)}e_{[n]}^{(1)} \\ &\quad \times z^\alpha (1 - z)^{\beta_{[n-j]}} F(a_{[n-j]}, b_{[n-j]}, c_{[n-j]}; z), \end{aligned} \quad (3.71)$$

$$\begin{aligned} \psi_{[n-j]}^{(2)} &= (-1)^j \prod_{u=0}^{j-1} \frac{2u - n + \frac{1}{2} + m - i(k_+ - k_-)}{2u - n - \frac{1}{2} + m + i(k_+ + k_-)}e_{[n]}^{(2)} \\ &\quad \times z^{\alpha+1/2} (1 - z)^{\beta_{[n-j]}} F(a_{[n-j]}, b_{[n-j]} + 1, c_{[n-j]} + 1; z). \end{aligned} \quad (3.72)$$

To obtain the quasi-normal modes we need to impose the vanishing Dirichlet condition at the boundary $z \rightarrow 1$. For the case $m > 0$, the dominant behaviour of the solutions near the boundary is given by

$$\begin{aligned} \psi_{[n-j]}^{(1)} &\simeq (-1)^j \prod_{u=0}^{j-1} \frac{2u - n + \frac{1}{2} + m - i(k_+ - k_-)}{2u - n + \frac{3}{2} + m + i(k_+ + k_-)} e_{[n]}^{(1)} \\ &\times (1 - z)^{-\beta_{[n-j]}} \frac{\Gamma(c_{[n-j]})\Gamma(a_{[n-j]} + b_{[n-j]} - c_{[n-j]})}{\Gamma(a_{[n-j]})\Gamma(b_{[n-j]})}, \end{aligned} \quad (3.73)$$

$$\begin{aligned} \psi_{[n-j]}^{(2)} &\simeq (-1)^j \prod_{u=0}^{j-1} \frac{2u - n + \frac{1}{2} + m - i(k_+ - k_-)}{2u - n - \frac{1}{2} + m + i(k_+ + k_-)} e_{[n]}^{(2)} \\ &\times (1 - z)^{-\beta_{[n-j]}} \frac{\Gamma(c_{[n-j]} + 1)\Gamma(a_{[n-j]} + b_{[n-j]} - c_{[n-j]})}{\Gamma(a_{[n-j]})\Gamma(b_{[n-j]} + 1)}. \end{aligned} \quad (3.74)$$

We can obtain the quasinormal modes by requiring that the coefficients of these leading terms vanish. Note that

$$a_{[n-j]} = a_{[n]} + j, \quad (3.75)$$

From this we have

$$\begin{aligned} \Gamma(a_{[n-j]}) &= \Gamma(a_{[n]} + j), \\ &= \Gamma(a_{[n]}) \prod_{u=0}^{j-1} (a_{[n]} + u), \\ &= \frac{\Gamma(a_{[n]})}{2^j} \prod_{u=0}^{j-1} (2u - n + \frac{1}{2} + m - i(k_+ - k_-)). \end{aligned} \quad (3.76)$$

The product $\prod_{u=0}^{j-1} (2u - n + \frac{1}{2} + m - i(k_+ - k_-))$ exactly cancels the numerator of the coefficient in (3.73) and (3.74). Thus the behaviour of the solutions near the boundary reduces to

$$\begin{aligned} \psi_{[n-j]}^{(1)} &\simeq \frac{(-2)^j}{\prod_{u=0}^{j-1} 2u - n + \frac{3}{2} + m + i(k_+ + k_-)} e_{[n]}^{(1)} \\ &\times (1 - z)^{-\beta_{[n-j]}} \frac{\Gamma(c_{[n-j]})\Gamma(a_{[n-j]} + b_{[n-j]} - c_{[n-j]})}{\Gamma(a_{[n]})\Gamma(b_{[n-j]})}, \end{aligned} \quad (3.77)$$

$$\begin{aligned} \psi_{[n-j]}^{(2)} &\simeq \frac{(-2)^j}{\prod_{u=0}^{j-1} 2u - n - \frac{1}{2} + m + i(k_+ + k_-)} e_{[n]}^{(2)} \\ &\times (1 - z)^{-\beta_{[n-j]}} \frac{\Gamma(c_{[n-j]} + 1)\Gamma(a_{[n-j]} + b_{[n-j]} - c_{[n-j]})}{\Gamma(a_{[n]})\Gamma(b_{[n-j]} + 1)}. \end{aligned} \quad (3.78)$$

The vanishing Dirichlet conditions at the boundary constrain the leading behaviour of all components of the tensorial spinor to vanish at infinity. As argued below

(3.38) the components which involve the radial coordinate are suppressed at the boundary compared to the components which only involved the boundary coordinates $+, -$. Thus it is sufficient to impose vanishing Dirichlet boundary conditions on these components. This is equivalent to looking at the common set of zeroes of the coefficients which occur in the functions given in (3.77). These are clearly given by

$$a_{[n]} = -\hat{n} \quad \text{and} \quad b_{[0]} + 1 = -\hat{n}, \quad \hat{n} = 0, 1, 2, \dots \quad (3.79)$$

In terms of k_+ and k_- these translate to

$$i(k_+ - k_-) = 2\hat{n} + \hat{\Delta} - s, \quad i(k_+ + k_-) = 2\hat{n} + \hat{\Delta} + s, \quad \hat{n} = 0, 1, 2, \dots, \quad (3.80)$$

where $s = n + \frac{1}{2}$. Thus the quasinormal modes are given by

$$\begin{aligned} \omega_L &= k + 2\pi T_L(k_+ + k_-), & \omega_R &= -k + 2\pi T_R(k_+ - k_-), \\ &= k - 2\pi i T_L(2\hat{n} + \hat{\Delta} + s), & &= -k - 2\pi i T_R(2\hat{n} + \hat{\Delta} - s). \end{aligned} \quad (3.81)$$

These coincide precisely with the poles of the corresponding two point function (1.3) for the corresponding spin s field as expected from the AdS/CFT correspondence. Reading out h_L and h_R we get $h_R - h_L = -s$, the case $h_R - h_L = +s$ will arise when we carry out the same analysis but with $m < 0$.

4. 1-loop determinant for arbitrary half-integer spins

As shown in [11], the poles of the retarded Green's function can be used to construct the one-loop determinant of the corresponding field in the bulk. In [11], the one-loop determinant for scalars in asymptotically *AdS* black holes including the BTZ black holes was constructed using analyticity and the information of the quasinormal modes. This construction was extended to arbitrary integer spins in [1]. In this section we would like to use the quasinormal mode of the higher spin fermion along with analyticity to construct the one loop determinant of the corresponding half integer spin field. We will then show that this determinant agrees with that constructed in [12] using group theoretic methods.

4.1 1-loop determinant from the spectrum of quasinormal modes

Our analysis will follow the method developed in [11, 12]. We consider the non-rotating BTZ black hole for which the metric is given by

$$ds^2 = -(r^2 - r_+^2)dt^2 + \frac{dr^2}{r^2 - r_+^2} + r^2 d\phi^2. \quad (4.1)$$

We then continue the BTZ black hole to Euclidean time together with the identification

$$t = -i\tau, \quad \tau \sim \tau + \frac{1}{T}, \quad (4.2)$$

and

$$T = T_H = T_L = \frac{r_+}{2\pi}. \quad (4.3)$$

Our goal to evaluate the following one loop determinant of the higher spin $s = n + \frac{1}{2}$ Laplacian

$$Z_s(\hat{\Delta}) = \det(-\nabla^2(s) + \mathcal{M}_s^2), \quad (4.4)$$

where \mathcal{M}_s is the mass shift which occurs on squaring the Dirac operator which acts on the higher spin fermion. The result of squaring the Dirac operator is given by

$$(\not{\nabla} + m_n)(\not{\nabla} - m_n)\Psi_{\mu_1\mu_2\dots\mu_n} = (\nabla^2 + (s+1) - m_n^2)\Psi_{\mu_1\mu_2\dots\mu_n} = 0. \quad (4.5)$$

where $s = n + \frac{1}{2}$. The proof of the above identity is given in Appendix A. Thus the mass \mathcal{M}_s is given by

$$\mathcal{M}_s^2 = m_n^2 - (s+1). \quad (4.6)$$

The basic strategy adopted by [11] to evaluate the determinant in (4.4) is to identify the zeros of the determinant in the complex $\hat{\Delta}$ space. Here $\hat{\Delta}$ is the conformal dimension of the corresponding dual operator. This occurs whenever the wave equation of the corresponding field has a zero mode and also obeys the periodicity (4.2). For the case of the spin s field these modes are given by

$$\omega_L \equiv \omega_L = k - 2\pi iT(2\hat{n} + \hat{\Delta} + s), \quad \omega_R \equiv \omega_R = -k - 2\pi iT(2\hat{n} + \hat{\Delta} - s). \quad (4.7)$$

where $\hat{n} = 0, 1, 2, \dots$ where k is the momentum along the ϕ direction. Thus it is quantized and therefore takes values in the set of integers

$$k = 0, \pm 1, \pm 2, \dots \quad (4.8)$$

Note that for the modes in (4.7) we have considered the situation when $h_L > h_R$. We now define

$$\begin{aligned} z_L &= k - 2\pi iT(2n + \hat{\Delta} + s), & \bar{z}_L &= k + 2\pi iT(2n + \hat{\Delta} + s), \\ z_R &= -k - 2\pi iT(2n + \hat{\Delta} - s), & \bar{z}_R &= -k + 2\pi iT(2n + \hat{\Delta} - s). \end{aligned} \quad (4.9)$$

Requiring the quasinormal modes to obey the thermal periodicity conditions due to the identification in (4.2) results in the following equations

$$\begin{aligned} 2\pi iT(\tilde{n} - s) &= z_L(\hat{\Delta}), & \tilde{n} &\geq 0, \\ 2\pi iT(\tilde{n} + s) &= \bar{z}_L(\hat{\Delta}), & \tilde{n} &< 0, \\ 2\pi iT(\tilde{n} + s) &= z_R(\hat{\Delta}), & \tilde{n} &\geq 0, \\ 2\pi iT(\tilde{n} - s) &= \bar{z}_R(\hat{\Delta}), & \tilde{n} &< 0. \end{aligned} \quad (4.10)$$

Note that since \tilde{n} is an integer $\frac{z_{L,R}(\hat{\Delta})}{2\pi iT}$ is half integral moded. These equalities imply that when $\hat{\Delta}$ is tuned to these values, the one loop determinant exhibits zeros. The ranges \tilde{n} are chosen so that the quantities

$$k - 2\pi iT(2n + \hat{\Delta}), \quad \text{and} \quad k + 2\pi iT(2n + \hat{\Delta}). \quad (4.11)$$

when considered together take values $2\pi iT\tilde{n}$ where \tilde{n} assumes values in the set of integers. Similarly the range of \tilde{n} for the case of the right-moving quasinormal modes is chosen so that the quantities

$$-k - 2\pi iT(2n + \hat{\Delta}), \quad \text{and} \quad -k + 2\pi iT(2n + \hat{\Delta}). \quad (4.12)$$

when considered together take values $2\pi iT\tilde{n}$ where \tilde{n} assumes values in the set of integers. Note that these are the same quantization conditions used in [1] for the case of integer spin s . Though we do not have a first principle justification of the choice of these ranges we will show that they do indeed lead to the answer evaluated using the group theoretic methods given in [12]. The function which is analytic in $\hat{\Delta}$ and has zeros at the locations (4.10) is given by

$$\begin{aligned} Z_{(s)} = e^{\text{Pol}(\hat{\Delta})} \prod_{\substack{z_L, \bar{z}_L \\ z_R, \bar{z}_R}} \left[\left(-s + \frac{iz_L}{2\pi T} \right) \left(s + \frac{iz_R}{2\pi T} \right) \right. \\ \times \prod_{v+\frac{1}{2} > -s} \left(v + \frac{1}{2} + \frac{iz_L}{2\pi T} \right) \left(v + \frac{1}{2} - \frac{i\bar{z}_L}{2\pi T} \right) \\ \times \left. \prod_{v+\frac{1}{2} > s} \left(v + \frac{1}{2} + \frac{iz_R}{2\pi T} \right) \left(v + \frac{1}{2} - \frac{i\bar{z}_R}{2\pi T} \right) \right]. \quad (4.13) \end{aligned}$$

where $\text{Pol}(\hat{\Delta})$ is a non-singular holomorphic function of $\hat{\Delta}$ and can be determined by examining the $\hat{\Delta} \rightarrow \infty$ behaviour. The product over $z_L, \bar{z}_L, z_R, \bar{z}_R$ mean the product over k, n occurring in the definition of these variables. The variable v takes values in the set of integers. Plugging in the quasi-normal modes into (4.13) and performing simple manipulations we obtain

$$\begin{aligned} Z_{(s)} = e^{\text{Pol}(\hat{\Delta})} \prod_{N \geq 0, p} \left[\left((2N + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right)^{-1} \right. \\ \times \left. \prod_{v \geq 0} \left((v + 2N + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right)^2 \right]. \quad (4.14) \end{aligned}$$

The analysis here is for the case $h_L - h_R = s$. On performing the evaluation for the one loop determinant for the situation with $h_R - h_L = -s$ it can be shown that one obtains the same expression as in (4.14). Since a higher spin fermion obeying the

second order spin s Laplacian contains both the modes we need to take the square of the expression in (4.14). Taking this into account and taking logarithms on both sides of the (4.14) we obtain

$$\begin{aligned}
-\log Z_{(s)} &= -\text{Pol}(\hat{\Delta}) - 4 \sum_{v \geq 0, N \geq 0, p} \log \left((v + 2N + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right) \\
&\quad + 2 \sum_{N \geq 0, p} \log \left((2N + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right), \\
&= -\text{Pol}(\hat{\Delta}) - 4 \sum_{v > 0, N \geq 0, p} \log \left((v + 2N + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right) \\
&\quad - 2 \sum_{N \geq 0, p} \log \left((2N + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right), \\
&= -\text{Pol}(\hat{\Delta}) - 2 \sum_{\kappa \geq 0, p} (\kappa + 1) \log \left((\kappa + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right). \tag{4.15}
\end{aligned}$$

The sums over v and N were combined and written as a sum over $\kappa = n + 2N$. We have also made use of $\log(a + ib) + \log(a - ib) = \log(a^2 + b^2)$. The divergent sums can then be extracted and absorbed in $\text{Pol}(\hat{\Delta})$. We then make use of the identity $\sum_{p \geq 1} \log \left(1 + \frac{x^2}{p^2} \right) = \log \frac{\sinh \pi x}{\pi x} = \pi x - \log(\pi x) + \log(1 - 2e^{-2\pi x})$ to obtain

$$\log Z_{(s)} = \text{Pol}(\hat{\Delta}) - 4 \log \prod_{\kappa \geq 0} (1 - q^{-\kappa + \hat{\Delta}})^{-(\kappa + 1)}, \tag{4.16}$$

where,

$$q = e^{2\pi i \tilde{\tau}}, \quad \tilde{\tau} = 2\pi i T. \tag{4.17}$$

To determine $\text{Pol}(\hat{\Delta})$ we use the same argument as in [11]. Note that taking the $\hat{\Delta} \rightarrow \infty$ the partition function should reduce to that of the BTZ which is locally identical to that of AdS_3 . This determines $\text{Pol}(\hat{\Delta})$ to be a function proportional to the volume of the Euclidean BTZ black hole. We will not write this explicitly since we do not require it in the subsequent discussion.

4.2 1-loop determinant from the heat kernel

We now show that the finite term in the one loop partition function (4.16) which is determined from the product over quasinormal modes agrees with that constructed from the heat kernel of the spin s field. The trace of the heat kernel for the spin s Laplacian on thermal AdS_3 is given by [12]

$$\text{Tr}(e^{-t\nabla_{(s)}^2}) = K^{(s)}(\tau, \bar{\tau}; t) = \sum_{n=1}^{\infty} \frac{\tau_2}{\sqrt{4\pi t} |\sin \frac{n\tau}{2}|^2} \cos(sn\tau_1) e^{-\frac{n^2 \tau_2^2}{4t}} e^{-(s+1)t}. \tag{4.18}$$

The formula retains only the finite term in the heat kernel and suppresses the term which is proportional to the volume of the AdS_3 . The parameter τ is related to the temperature of the Euclidean non-rotating BTZ by

$$\tau = \frac{i}{2\pi T}. \quad (4.19)$$

We will substitute this value of τ at the end of our analysis. The 1-loop determinant is then given by

$$-\log(\det(-\nabla^2 + \mathcal{M}_s^2)) = \int_0^\infty \frac{dt}{t} e^{-\mathcal{M}_s^2 t} K^{(s)}(\tau, \bar{\tau}; t), \quad (4.20)$$

where \mathcal{M}_s is given in (4.6). Substituting the value of \mathcal{M}_s we obtain

$$e^{-\mathcal{M}_s^2 t} K^{(s)}(\tau, \bar{\tau}; t) = \sum_{u=1}^{\infty} \frac{\tau_2}{\sqrt{4\pi t} |\sin \frac{u\tau}{2}|^2} \cos(su\tau_1) e^{-\frac{u^2 \tau_2^2}{4t}} e^{-m_n^2 t}. \quad (4.21)$$

The integration over t can be performed as follows

$$\frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{dt}{t^{3/2}} e^{-\frac{u^2 \tau_2^2}{4t}} e^{-m_n^2 t} = \frac{1}{u\tau_2} e^{-u\tau_2 m_n} = e^{-u\tau_2(\hat{\Delta}-1)}. \quad (4.22)$$

Here we have used the relation $\hat{\Delta} = 1 + m = 1 + m_n$ derived in (3.40). Thus the one loop determinant reduces to

$$\begin{aligned} -\log(\det(-\nabla^2 + \mathcal{M}_s^2)) &= \sum_{u=1}^{\infty} \frac{\cos(su\tau_1)}{u |\sin \frac{u\tau}{2}|^2} e^{-u\tau_2(\hat{\Delta}-1)}, \\ &= \sum_{u=1}^{\infty} \frac{2}{u} \frac{(q^{su} + \bar{q}^{su})}{|1 - q^u|^2} q^{(\hat{\Delta}-s)u}, \\ &= \sum_{u=1}^{\infty} \frac{4}{u} \frac{q^{\hat{\Delta}u}}{(1 - q^u)^2}, \\ &= -4 \log \prod_{v=0}^{\infty} (1 - q^{v+\hat{\Delta}})^{m+1}. \end{aligned} \quad (4.23)$$

where

$$q = e^{2\pi i \tau}. \quad (4.24)$$

In the above manipulations we have also used the fact that τ is purely imaginary for the case of the non-rotating BTZ black hole which results in $q = \bar{q}$. Thus we obtain

$$\log Z_{(s)} = \log(\det(-\nabla^2 + \mathcal{M}_s^2)) = 4 \log \prod_{v=0}^{\infty} (1 - q^{v+\hat{\Delta}})^{-(m+1)}. \quad (4.25)$$

Comparing (4.16) and (4.25) we see that the two expressions indeed agree on performing the modular transformation

$$\tilde{\tau} = -\frac{1}{\tau}. \quad (4.26)$$

which is the expected relation between one-loop determinants on Euclidean BTZ and thermal AdS_3 .

5. Conclusions

We have solved the wave equations for arbitrary massive higher spin fermionic fields in the BTZ background. In this work we focused on the ingoing modes at the horizon to obtain the quasi-normal modes, but the analysis can be easily extended for the outgoing modes. This will lead to the complete set of modes for the higher spin fermion in this background which is the starting point for its quantization. This can be useful for studying fermionic emission by Hawking radiation on similar lines as [18]. It will also be useful to find the exact prescription to evaluate the retarded Green's function for higher spin fermions extending the work done for the spin 1/2 and spin 3/2 cases in [19] and [20] respectively.

From our discussion of the wave equations for massive higher spin fermionic fields it is clear that other properties like the bulk to boundary propagator for these fields can also be solved and obtained in closed form. These are important tools to study the general AdS_3/CFT_2 correspondence and it is useful to obtain them. Finally the work in this paper and that related to massive higher spin integer spins in [1] and the observation that classical string propagation in BTZ is integrable [2] suggests that it is possible to quantize strings in the BTZ background.

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A. The background geometry

The BTZ black hole

The metric of the BTZ black hole is conventionally written as

$$ds^2 = -\frac{\Delta^2}{r^2}dt^2 + \frac{r^2}{\Delta^2}dr^2 + r^2 \left(d\phi - \frac{r_+r_-}{r^2}dt \right)^2, \quad (A.1)$$
$$\Delta^2 = (r^2 - r_+^2)(r^2 - r_-^2).$$

Here r_+ and r_- are the radii of the inner and outer horizons respectively, r is the radial distance and t labels the time. The angular coordinate ϕ has the period of 2π . We will work with units in which the radius of AdS_3 is unity. The left and right temperatures are defined as

$$T_L = \frac{1}{2\pi}(r_+ - r_-), \quad T_R = \frac{1}{2\pi}(r_+ + r_-). \quad (A.2)$$

A convenient coordinate system for our analysis was discovered by [7]. We first define the coordinates

$$z = \tanh^2 \xi = \frac{r_+^2 - r_-^2}{r_+^2 + r_-^2}, \quad (\text{A.3})$$

$$x^+ = r_+ t - r_- \phi, \quad x^- = r_+ \phi - r_- t.$$

Note that in these coordinates, the range of r from r_+ to ∞ is mapped to $z = 0$ or $\xi = 0$ to $z = 1$ or $\xi = \infty$ respectively. In these coordinates, the BTZ metric given in (A.1) reduces to the following diagonal metric

$$ds^2 = d\xi^2 - \sinh^2 \xi dx_+^2 + \cosh^2 \xi dx_-^2. \quad (\text{A.4})$$

This form of the metric is used in our calculations and we will briefly list its various properties. The non-vanishing Christoffel symbols of the metric in (A.4) are given by

$$\begin{aligned} \tilde{\Gamma}_{++}^\xi &= \cosh \xi \sinh \xi = \frac{\sqrt{z}}{1-z}, & \tilde{\Gamma}_{--}^\xi &= -\cosh \xi \sinh \xi = -\frac{\sqrt{z}}{1-z}, \\ \tilde{\Gamma}_{+\xi}^+ &= \coth \xi = \frac{1}{\sqrt{z}}, & \tilde{\Gamma}_{-\xi}^- &= \tanh \xi = \sqrt{z}. \end{aligned}$$

The metric and its Christoffel symbols obey the following identities which will be useful in simplifying the higher spin equations in the next sections

$$\sqrt{-g} = \cosh \xi \sinh \xi = \frac{\sqrt{z}}{1-z}, \quad (\text{A.5})$$

$$\frac{g_{++}}{\sqrt{-g}} = -\tanh \xi = -\sqrt{z}, \quad (\text{A.6})$$

$$\frac{g_{--}}{\sqrt{-g}} = \coth \xi = \frac{1}{\sqrt{z}}, \quad (\text{A.7})$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \tilde{\Gamma}_{\nu\rho}^\sigma) = \frac{1}{\sqrt{-g}} \partial_\xi (\sqrt{-g} g^{\xi\xi} \tilde{\Gamma}_{\xi\rho}^\sigma) = 2\hat{\delta}_\rho^\sigma \quad (\text{A.8})$$

$$g^{\pm\pm} \tilde{\Gamma}_{\pm\pm}^\xi + \tilde{\Gamma}_{\xi\pm}^\pm = 0, \quad (\text{A.9})$$

$$g^{++} \tilde{\Gamma}_{++}^\xi = -\coth \xi = -\frac{1}{\sqrt{z}}, \quad (\text{A.10})$$

$$g^{--} \tilde{\Gamma}_{--}^\xi = -\tanh \xi = -\sqrt{z}. \quad (\text{A.11})$$

where $\hat{\delta}_{\rho\sigma}$ is defined as

$$\hat{\delta}_\rho^\sigma = \begin{cases} 1 & \text{for } \rho, \sigma = \pm \text{ and } \rho = \sigma, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A.12})$$

The BTZ black hole is obtained by identifications of AdS_3 [17]. Thus it is locally AdS_3 and therefore its curvature obey the following relations

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\delta} g_{\beta\gamma} - g_{\alpha\gamma} g_{\beta\delta}, \quad (\text{A.13})$$

$$R_{\mu\nu} = -2g_{\mu\nu}, \quad G_{\mu\nu} = g_{\mu\nu}. \quad (\text{A.14})$$

In 3 dimensions, the Riemann tensor further obeys the following relation

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \epsilon_{\alpha\beta\rho}\epsilon_{\gamma\delta\sigma}(R^{\rho\sigma} - \frac{1}{2}Rg^{\rho\sigma}), \\ &= \epsilon_{\alpha\beta\rho}\epsilon_{\gamma\delta\sigma}G^{\rho\sigma}. \end{aligned} \quad (\text{A.15})$$

Here $G^{\rho\sigma}$ is the Einstein tensor and the epsilon tensor is defined as

$$\epsilon^{\alpha\beta\gamma} = \frac{\tilde{\epsilon}^{\alpha\beta\gamma}}{\sqrt{-g}}, \quad \tilde{\epsilon}^{+\xi-} = 1. \quad (\text{A.16})$$

where, $\tilde{\epsilon}^{\alpha\beta\gamma}$ is the completely antisymmetric Levi-Civita symbol. The epsilon tensor in 3 dimensions satisfies the relation

$$\epsilon_{\beta\rho}{}^{\alpha}\epsilon_{\alpha\delta\sigma} = -(g_{\beta\delta}g_{\rho\sigma} - g_{\beta\sigma}g_{\rho\delta}). \quad (\text{A.17})$$

Now using the definition of the flat space gamma matrices given in (2.20) and the vierbein in (2.21) it is easy to see that the the curved space gamma matrices satisfy

$$[\Gamma^{\mu}, \Gamma^{\nu}] = 2\epsilon^{\mu\nu\rho}\Gamma_{\rho}, \quad \Gamma^{\mu}\Gamma_{\mu} = 3. \quad (\text{A.18})$$

Relation between ∇^2 and ∇^2 in AdS_3

To establish the relation between the spin s Laplacian and ∇^2 we start by considering the action of ∇^2 on the object $\Psi_{\mu_1\mu_2\ldots\mu_n}$.

$$\begin{aligned} \nabla^2 \Psi_{\mu_1\mu_2\ldots\mu_n} &= \Gamma^{\mu}\Gamma^{\nu}\nabla_{\mu}\nabla_{\nu}\Psi_{\mu_1\mu_2\ldots\mu_n}, \\ &= \frac{1}{2}\{\Gamma^{\mu}, \Gamma^{\nu}\}\nabla_{\mu}\nabla_{\nu}\Psi_{\mu_1\mu_2\ldots\mu_n} + \frac{1}{2}[\Gamma^{\mu}, \Gamma^{\nu}]\nabla_{\mu}\nabla_{\nu}\Psi_{\mu_1\mu_2\ldots\mu_n}, \\ &= \nabla^2 \Psi_{\mu_1\mu_2\ldots\mu_n} + \frac{1}{4}[\Gamma^{\mu}, \Gamma^{\nu}][\nabla_{\mu}, \nabla_{\nu}]\Psi_{\mu_1\mu_2\ldots\mu_n}. \end{aligned} \quad (\text{A.19})$$

Thus, we require to evaluate the second term in the above expression.

$$\begin{aligned} &\frac{1}{4}[\Gamma^{\mu}, \Gamma^{\nu}][\nabla_{\mu}, \nabla_{\nu}]\Psi_{\mu_1\mu_2\ldots\mu_n} \\ &= \frac{1}{32}R_{\mu\nu\sigma\delta}[\Gamma^{\mu}, \Gamma^{\nu}][\Gamma^{\sigma}, \Gamma^{\delta}]\Psi_{\mu_1\mu_2\ldots\mu_n} \\ &\quad + \frac{1}{4}[\Gamma^{\mu}, \Gamma^{\nu}]g^{\eta\alpha}(R_{\alpha\mu_1\nu\mu}\Psi_{\eta\mu_2\ldots\mu_n} + R_{\alpha\mu_2\nu\mu}\Psi_{\mu_1\eta\mu_3\ldots\mu_n} + \cdots + R_{\alpha\mu_n\nu\mu}\Psi_{\mu_1\mu_2\ldots\mu_{n-1}\eta}), \\ &= \frac{1}{32}\mathcal{G}_{\mu_1\mu_2\ldots\mu_n} + \frac{1}{4}\mathcal{H}_{\mu_1\mu_2\ldots\mu_n}. \end{aligned} \quad (\text{A.20})$$

To obtain the first equality we have used the definition of the covariant derivative given in (2.6). We then define

$$\mathcal{G}_{\mu_1\mu_2\ldots\mu_n} = R_{\mu\nu\sigma\delta}[\Gamma^{\mu}, \Gamma^{\nu}][\Gamma^{\sigma}, \Gamma^{\delta}]\Psi_{\mu_1\mu_2\ldots\mu_n}, \quad (\text{A.21})$$

$$\begin{aligned} \mathcal{H}_{\mu_1\mu_2\ldots\mu_n} &= [\Gamma^{\mu}, \Gamma^{\nu}]g^{\eta\alpha}(R_{\alpha\mu_1\nu\mu}\Psi_{\eta\mu_2\ldots\mu_n} + R_{\alpha\mu_2\nu\mu}\Psi_{\mu_1\eta\mu_3\ldots\mu_n} + \cdots \\ &\quad + R_{\alpha\mu_n\nu\mu}\Psi_{\mu_1\mu_2\ldots\mu_{n-1}\eta}). \end{aligned} \quad (\text{A.22})$$

Let's evaluate $\mathcal{G}_{\mu_1\mu_2\ldots\mu_n}$ first.

$$\begin{aligned}
\mathcal{G}_{\mu_1\mu_2\ldots\mu_n} &= R_{\mu\nu\sigma\delta}[\Gamma^\mu, \Gamma^\nu][\Gamma^\sigma, \Gamma^\delta]\Psi_{\mu_1\mu_2\ldots\mu_n}, \\
&= (g_{\mu\delta}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\delta})(2\epsilon^{\mu\nu\alpha}\Gamma_\alpha)(2\epsilon^{\sigma\delta\beta}\Gamma_\beta)\Psi_{\mu_1\mu_2\ldots\mu_n}, \\
&= 8(\delta_\eta^\eta g^{\alpha\beta}\Gamma_\alpha\Gamma_\beta - \delta_\eta^\alpha g^{\eta\beta}\Gamma_\alpha\Gamma_\beta)\Psi_{\mu_1\mu_2\ldots\mu_n}, \\
&= 8(3 \cdot 3 - 3)\Psi_{\mu_1\mu_2\ldots\mu_n} = 48 \Psi_{\mu_1\mu_2\ldots\mu_n}.
\end{aligned} \tag{A.23}$$

In performing these manipulations we have used the identities in (A.17) and (A.18). Let's now evaluate $\mathcal{H}_{\mu_1\mu_2\ldots\mu_n}$.

$$\begin{aligned}
&\mathcal{H}_{\mu_1\mu_2\ldots\mu_n} \\
&= [\Gamma^\mu, \Gamma^\nu]g^{\eta\alpha}(R_{\alpha\mu_1\nu\mu}\Psi_{\eta\mu_2\ldots\mu_n} + R_{\alpha\mu_2\nu\mu}\Psi_{\mu_1\eta\mu_3\ldots\mu_n} + \cdots + R_{\alpha\mu_n\nu\mu}\Psi_{\mu_1\mu_2\ldots\mu_{n-1}\eta}), \\
&= g^{\alpha\eta}[\Gamma^\mu, \Gamma^\nu]\sum_{j=1}^n(g_{\alpha\mu}g_{\mu_j\nu} - g_{\alpha\nu}g_{\mu_j\mu})\Psi_{\eta\mu_1\ldots\check{\mu}_j\ldots\mu_n}, \\
&= 2\Gamma^\mu\Gamma^\nu\sum_{j=1}^ng_{\mu_j\nu}\Psi_{\mu\mu_1\ldots\check{\mu}_j\ldots\mu_n} = 2(2g^{\mu\nu} - \Gamma^\nu\Gamma^\mu)\sum_{j=1}^ng_{\mu_j\nu}\Psi_{\mu\mu_1\ldots\check{\mu}_j\ldots\mu_n}, \\
&= 4g^{\mu\nu}\sum_{j=1}^ng_{\mu_j\nu}\Psi_{\mu\mu_1\ldots\check{\mu}_j\ldots\mu_n} = 4\sum_{j=1}^n\Psi_{\mu_1\ldots\mu_j\ldots\mu_n}, \\
&= 4n\Psi_{\mu_1\mu_2\ldots\mu_n}.
\end{aligned} \tag{A.24}$$

Here we have used the tracelessness condition (2.3) several times to simplify the terms. Using (A.23) and (A.24) in (A.20) we have

$$\begin{aligned}
\frac{1}{4}[\Gamma^\mu, \Gamma^\nu][\nabla_\mu, \nabla_\nu]\Psi_{\mu_1\mu_2\ldots\mu_n} &= \frac{48}{32}\Psi_{\mu_1\mu_2\ldots\mu_n} + \frac{4n}{4}\Psi_{\mu_1\mu_2\ldots\mu_n}, \\
&= \left(n + \frac{3}{2}\right)\Psi_{\mu_1\mu_2\ldots\mu_n}, \\
&= (s + 1)\Psi_{\mu_1\mu_2\ldots\mu_n}.
\end{aligned} \tag{A.25}$$

Thus from (A.19) we have

$$\nabla^2\Psi_{\mu_1\mu_2\ldots\mu_n} = (\nabla^2 + (s + 1))\Psi_{\mu_1\mu_2\ldots\mu_n}. \tag{A.26}$$

B. Diagonalization of the mass matrix: Proof

In this section we will prove Identity 2 (3.22) which is given by

$$T^{(n)}M^{(n)}[T^{(n)}]^{-1} = D^{(n)}, \tag{B.1}$$

$$\begin{aligned}
M^{(n)}[T^{(n)}]^{-1} &= [T^{(n)}]^{-1}D^{(n)}, \\
M^{(n)}T^{(n)} &= T^{(n)}D^{(n)}.
\end{aligned} \tag{B.2}$$

where, $D_{pq}^{(n)} = (n - 2p + m)\delta_{pq}$. To arrive at the last line we have used Identity 1 (3.21). This is equivalent to showing

$$\sum_{b=0}^n \sum_{c=0}^n M_{ab}^{(n)} T_{bc}^{(n)} x^c = \sum_{b=0}^n \sum_{c=0}^n T_{ab}^{(n)} D_{bc}^{(n)} x^c. \quad (\text{B.3})$$

Let's start with the LHS of (B.3). and define the variable

$$z = \frac{x+1}{x-1}. \quad (\text{B.4})$$

Then we obtain

$$\begin{aligned} & \sum_{b=0}^n M_{ab}^{(n)} \sum_{c=0}^s T_{bc}^{(n)} x^c \\ &= \sum_{b=0}^n M_{ab}^{(n)} (x+1)^b (x-1)^{n-b}, \\ &= (x-1)^n \sum_{b=0}^n M_{ab}^{(n)} z^b, \\ &= (x-1)^n \sum_{b=0}^n [-a\delta_{a-1,b} - (n-a)\delta_{a+1,b} + m\delta_{a,b}] z^b, \\ &= (x-1)^n [-az^{a-1} - (n-a)z^{a+1} + mz^a], \\ &= (x-1)^{n-a-1} (x+1)^{a-1} [(m-n)x^2 + 2(2a-n)x - (m+n)]. \end{aligned} \quad (\text{B.5})$$

We now need to evaluate the RHS. Before proceeding we shall list a few definitions and identities which we shall be using.

$$\begin{aligned} P_{(n,a)}(x) &= \sum_{b=0}^n T_{ab}^{(n)} x^b = (x+1)^a (x-1)^{n-a}, \\ x \frac{dP_{(n,a)}(x)}{dx} &= \sum_{b=0}^n b T_{ab}^{(n)} x^b. \end{aligned} \quad (\text{B.6})$$

For the RHS we have,

$$\begin{aligned}
& \sum_{b=0}^n T_{ab}^{(n)} \sum_{c=0}^n D_{bc}^{(n)} x^c, \\
&= \sum_{b=0}^n T_{ab}^{(n)} \sum_{c=0}^n [n - 2b + m] \delta_{bc} x^c, \\
&= \sum_{b=0}^n T_{ab}^{(n)} [n - 2b + m] x^b, \\
&= (n + m) \sum_{b=0}^n T_{ab}^{(n)} x^b - 2 \sum_{b=0}^n b T_{ab}^{(n)} x^b, \\
&= (n + m) P_{(n,a)}(x) - 2x \frac{dP_{(n,a)}(x)}{dx}, \\
&= (n + m)(x + 1)^a (x - 1)^{n-a} \\
&\quad - 2x[a(x + 1)^{a-1}(x - 1)^{n-a} + (n - a)(x + 1)^a(x - 1)^{n-a-1}], \\
&= (x - 1)^{n-a-1}(x + 1)^{a-1}[(m - n)x^2 + 2(2a - n)x - (m + n)]. \tag{B.7}
\end{aligned}$$

This completes the proof of (3.22).

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